Combining House Price Indices in Temporal Hierarchies*

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Abstract

There is a growing demand from central banks, governments, banks, real estate developers, and households for reliable and more timely house price indices. Hill et al (2018) estimated a number of weekly hedonic imputed indices and found there are larger discrepancies in the resulting indices obtained from alternative modelling strategy at this higher frequency than that found when the index is constructed for lower frequencies (e.g. quarterly). Hyndman et al. (2011) and Athanasopoulos et al. (2017) develop a method for optimally combining time series in temporal hierarchies. In this paper we extend this approach to the case of price indices. In this case we replace the assumption that the time series at the top of the hierarchy is equal to the sum of the bottom time series in the hierarchy with the requirement that price indices of different frequencies should generate results that are on average equal at the frequency of the lowest frequency index. A number of alternative weighting schemes are considered, and a comparison is drawn with the multilateral price index literature. The empirical example presents temporal hierarchies for hedonic property price indices in Sydney, Australia. The

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frequency of the indices considered are annual, quarterly, monthly, and bimonthly. The two latter frequencies are of special interest to central banks. For example, the ECB meets bimonthly, while the Reserve Bank of Australia meets monthly. Real time high frequency indices can provide a timely indication of the state of the real estate market. We consider hedonic imputed indices constructed from semilog hedonic models which control for property location using either postcode dummies or a non-parametric geo-spline, and construct volatility measures to compare the unreconciled and reconciled indices. The results presented are preliminary.

1 Introduction

In recent years there is an increased interest in higher frequency house price indices. For example, since the global financial crisis central banks have become more aware of how developments in the housing market can affect the rest of the economy, and in some cases threaten financial stability. Hence there is a need for more timely data, thus allowing central banks and other regulatory bodies to respond faster to undesirable developments in the housing market, other asset markets, and more generally to macroeconomic variables such as inflation.

Nevertheless, there is still also a need for lower frequency indices that capture more long-run trends that are less affected by seasonal and other short-term trends. Inconsistencies however are often observed between higher and lower frequency indices. This raises the question of how such inconsistencies should be dealt with?

One solution is to take a top-down approach and assume the lowest frequency index is correct. Hence the higher frequency indices are adjusted until they are reconciled with the lowest frequency index. Alternatively, a bottom-up approach assumes the highest frequency index is correct, and then constructs the lower frequency indices directly from the highest frequency index.

We consider here the case of price indices arranged in temporal hierarchies. In such a hierarchy, the basic building blocks are the time periods over which the highest frequency index is defined (e.g., quarters). The second highest frequency consists of a whole number of highest frequency periods (e.g., two quarters). The next highest frequency consists of a whole number of periods from the previous layer in teh hierarchy (e.g., two half years), etc.

Rather than viewing such inconsistencies as a nuisance, they provide an opportunity to improve the quality of the price indices at all frequencies. With this in mind, we propose here a least squares method that reconciles price indices arranged in temporal hierarchies. It follows that all the indices are adjusted in the reconciliation process.

Our method is related to a least squares reconciliation approach for temporal hierarchies developed by Athanasopoulos, Hyndman, Kourentzes, and Petropoulos (2017), which in turn draws on Hyndman, Ahmed, Athanasopoulos, Shang (2011), and Hyndman, Lee and Wang (2016). There is, however, an important difference between our approach and that of Athanasopoulos et al. (2017). Athanasopoulos et al. focus on data series that can be summed across time periods. In other words, it is assumed that the annual value of a series in a particular year should equal the sum of the quarterly series in that same year.

We do not make such an assumption. Our focus here is on price indices. When expressed in log form, at first glance one might think that such an assumption is reasonable. This would imply that a quarterly chained price index from the first quarter in year 1 to the last quarter in year 2 should equal an annual price index between years 1 and 2, which can be expressed algebraically as follows:

$$\ln P_{1q1,1q2}+\ln P_{1q2,1q3}+\ln P_{1q3,1q4}+\ln P_{1q4,2q1}+\ln P_{2q1,2q2}+\ln P_{2q2,1q3}+\ln P_{2q3,2q4}=\ln P_{1,2}.$$

Here $P_{1q1,1q2}$ denotes a price index comparing year 1-quarter 1 and year 1-quarter 2, while $P_{1,2}$ denotes a price index comparing years 1 and 2. P equals 1 if there is no price change. If prices rise by say 2 percent, then P equals 1.02.

Our contention, however, is that such a restriction is not justified. The price index comparing years 1 and 2 effectively measures the average change in prices from years 1 and 2. The chained quarterly price index measures the change in prices from the first quarter in year 1 to the fourth quarter in year 2. For example, suppose prices rise steadily throughout the sample period. It follows that the chained quarterly index should be larger than the annual index, since the annual index does not account for price rises during each year, rather focusing on price changes between the two years.

Instead of assuming that the price indices in log form are additive over the temporal hierarchy, we impose different identifying restrictions. Our approach draws on ideas from the multilateral price index literature, and especially the Gini-Eltetö-Szulc (GEKS) method (see for example Diewert 1999 and Balk 2008). The key is to formulate different combinations of indices that provide alternative answers to the same question. In the example discussed above, an index equivalent to the annual index $\ln P_{1,2}$ can be obtained from the quarterly data as follows:

$$\ln P_{1,2}=\left(\frac{1}{4}\right)(\ln P_{1q1,2q1}+\ln P_{1q2,2q2}+\ln P_{1q3,2q3}+\ln P_{1q4,2q4}).$$

Each index on the right-hand-side of this equation uses quarterly data to measure price changes at an annual frequency. For example, $P_{1q1,2q1}$ measures the change in prices from quarter 1 in year 1 to quarter 1 in year 2, $P_{1q2,2q2}$ measures the change in prices from quarter 2 in year 1 to quarter 2 in year 2, etc. Averaging these four indices, we obtain an estimate of the annual price change from year 1 to 2, which can be compared with $P_{1,2}$. Our identifying restrictions entail requiring after reconciliation that these different indices asking the same question give the same answer.

The remainder of the paper is structured as follows. Section 2 develops our least squares reconciliation method as it applies to two and three layer temporal hierarchies. Section 3 interprets the reconciled price indices obtained from three layer hierarchies. Section 4 explores the link between our least squares reconciliation method for temporal hierarchies and the GEKS multilateral price index method which is used for reconciling intransitive bilateral spatial price indices. Section 5 considers weighted variants on our basic method. Section 6 provides an empirical application, using house price indices computed at a bimonthly, monthly, quarterly and annual frequency and proposes a possible metric to show the benefits of reconciliation. Our main findings are summarized in Section 7.

2 Reconciling Temporal Hierarchies of Price Indices

2.1 The simplest case

The simplest case of a temporal hierarchy of price indices is where there are two layers, and the higher frequency is double that of the lower frequency. Here we focus on the case where the lower frequency is annual and the higher frequency biannual. The reconciliation is done at the level of the lowest frequency index, which is here annual. We reconcile each pair of adjacent years separately.

In this simplest case we have three distinct price indices defined on an annual time horizon. Let $P_{1,2}$ denote the price change from year 1 and 2. $P_{11,12}$ the price change from the first half of year 1 to the first half of year 2, and $P_{21,22}$ the price change from the second half of year 1 to the second half of year 2. Taking the geometric mean of $P_{11,21}$ and $P_{12,22}$, we obtain an alternative measure to $P_{1,2}$ of the price change from year 1 to year 2.

Our objective is to alter the original indices $P_{1,2}$, $P_{11,21}$ and $P_{12,22}$ by the logarithmic least squares amount necessary to reconcile our two annualized indices. Reconciliation here means ensuring that the following condition is satisfied:

$\ln \hat{P}_{1,2} = 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22}))$.

This least squares problem can be formulated as follows:

$$\text{Min}_{\ln \hat{P}_{1,2}, \ln \hat{P}_{11,21}, \ln \hat{P}_{12,22}} [(\ln \hat{P}_{1,2} - \ln P_{1,2})^2 + 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22} - \ln P_{11,21} - \ln P_{12,22})^2],$$

$$\text{such that} \quad \ln \hat{P}_{1,2} = 0.5(\ln \hat{P}_{11,21} + \ln \hat{P}_{12,22}). \tag{1}$$

We can rewrite this problem more compactly in matrix notation as follows:

$$y = S\beta + \varepsilon,$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} \ln P_{1,2} \\ 0.5(\ln P_{11,21}) \\ 0.5(\ln P_{12,22}), \end{pmatrix}$$

and ε is an error vector. The projection matrix in this case is

$$S(S'S)^{-1}S' = \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}.$$

The least squares solution is now given by:

$$\hat{y} = S\hat{\beta} = S(S'S)^{-1}S'y, \tag{2}$$

where

$$\hat{y} = \begin{pmatrix} \ln \hat{P}_{1,2} \\ 0.5(\ln \hat{P}_{11,21}) \\ 0.5(\ln \hat{P}_{12,22}) \end{pmatrix} = \begin{pmatrix} \hat{\beta}_1 + \hat{\beta}_2 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2\ln P_{1,2} + \frac{1}{2}(\ln P_{11,21} + \ln P_{12,22}) \\ \ln P_{11,21} + \frac{1}{2}(2\ln P_{1,2} - \ln P_{12,22}) \\ \ln P_{12,22} + \frac{1}{2}(2\ln P_{1,2} - \ln P_{11,21}) \end{pmatrix}. \tag{3}$$

It follows from (3) that

$$\ln \hat{P}_{1,2} = \frac{1}{3}\left[2\ln P_{1,2} + \frac{1}{2}(\ln P_{11,21} + \ln P_{12,22})\right], \tag{4}$$

$$\ln \hat{P}_{11,21} = \frac{1}{3}\left[2\ln P_{11,21} + (2\ln P_{1,2} - \ln P_{12,22})\right], \tag{5}$$

$$\ln \hat{P}_{12,22} = \frac{1}{3}\left[2\ln P_{12,22} + (2\ln P_{1,2} - \ln P_{11,21})\right]. \tag{6}$$

From these equations in each case we can interpret the reconciled price index as a weighted geometric mean of the direct unreconciled index and the indirect unreconciled index, where the direct index is given by twice the weight of the indirect index.

2.2 Three layer hierarchies

Consider now the case of a temporal hierarchy consisting of annual, biannual, and quarterly price indices. Focusing on the reconciliation of years 1 and 2, we now have the following additional indices defined on an annual time interval: $P_{1q1,2q1}$ compares the first quarters of years 1 and 2, $P_{1q2,2q2}$ compares the second quarters of years 1 and 2, etc.

Now we have three reconciliation equations:

(i) $\hat{P}_{1,2} = (\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2} \times \hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/4}$

(ii) $(\hat{P}_{11,21}) = (\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2})^{1/2}$

(iii) $(\hat{P}_{12,22}) = (\hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/2}$

Three more equations relating the reconciled prices indices can be derived from (i), (ii) and (iii). These are the following:
(iv) $\hat{P}_{1,2} = (\hat{P}_{11,21} \times \hat{P}_{12,22})^{1/2}$.
(v) $\hat{P}_{1,2} = [(\hat{P}_{1q1,2q1} \times \hat{P}_{1q2,2q2})^{1/2} \times \hat{P}_{12,22}]^{1/2}$.
(vi) $\hat{P}_{1,2} = [\hat{P}_{11,21} \times (\hat{P}_{1q3,2q3} \times \hat{P}_{1q4,2q4})^{1/2}]^{1/2}$.

Our objective is to alter the unreconciled price indices by the logarithmic least squares amount necessary so that (i), (ii), and (iii) are satisfied. This reconciliation problem can be formulated in matrix notation as follows:

$$y = S\beta + \varepsilon,$$

where

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad y = \begin{pmatrix} \ln P_{1,2} \\ 0.5(\ln P_{11,21}) \\ 0.5(\ln P_{12,22}) \\ 0.25(\ln P_{1q1,2q1}) \\ 0.25(\ln P_{1q2,2q2}) \\ 0.25(\ln P_{1q3,2q3}) \\ 0.25(\ln P_{1q4,2q4}) \end{pmatrix}, \tag{7}$$

and ε again denotes an error vector.

3 Interpreting Reconciled Price Indices in Three Layer Hierarchies

It is informative to consider how the reconciled price indices are formed by taking linear combinations of the original unreconciled indices. In this regard we will focus on the three layer hierarchy of yearly, biannual, and quarterly indices. In this case, the matrix $S(S'S)^{-1}S'$ takes the following form:

$$S(S'S)^{-1}S' = \frac{1}{21}\begin{pmatrix} 12 & 6 & 6 & 3 & 3 & 3 & 3 \\ 6 & 10 & -4 & 5 & 5 & -2 & -2 \\ 6 & -4 & 10 & -2 & -2 & 5 & 5 \\ 3 & 5 & -2 & 13 & -8 & -1 & -1 \\ 3 & 5 & -2 & -8 & 13 & -1 & -1 \\ 3 & -2 & 5 & -1 & -1 & 13 & -8 \\ 3 & -2 & 5 & -1 & -1 & -8 & 13 \end{pmatrix} \tag{8}$$

Solutions for the reconciled annual price indices as functions of the original price indices are obtained by inserting (8) and the y vector in (7) into (2).

$$\ln \hat{P}_{1,2} = \frac{1}{21}\left\{12 \ln P_{1,2} + 6\left[\frac{1}{2}(\ln P_{11,21} + \ln P_{12,22})\right]\right.$$

$$+3\left[\frac{1}{4}(\ln P_{1q1,2q1}+\ln P_{1q2,2q2}+\ln P_{1q3,2q3}+\ln P_{1q4,2q4})\right]\Bigg\}. \tag{9}$$

It can be seen that the annual reconciled index is a weighted geometric mean of three competing unreconciled annualized indices. The direct unreconciled annual index $P_{1,2}$ gets a weight of 12/21. From reconciliation equation (iv), the indirect annual index obtained by taking the geometric mean of the two annualized biannual indices gets a weight of 6/21. Finally, substituting the reconciliation equations (ii) and (iii) into (iv), the indirect annual index obtained by taking the geometric mean of the four annualized quarterly indices gets a weight of 3/21.

The solution for $P_{11,21}$ is:

$$\ln \hat{P}_{11,21} = \frac{1}{21}\Bigg\{12\ln P_{1,2}+10\ln P_{11,21}-4\ln P_{12,22}+\frac{5}{2}\ln P_{1q1,2q1}+\frac{5}{2}\ln P_{1q2,2q2}$$
$$-\ln P_{1q3,2q3}-\ln P_{1q4,2q4}.\} \tag{10}$$

This solution for $P_{11,21}$ can be reinterpreted as follows:

$$\ln \hat{P}_{11,21} = \frac{1}{21}\Bigg\{10\ln P_{11,21}+4(2\ln P_{1,2}-\ln P_{12,22})+2\left[2\ln P_{1,2}-\frac{1}{2}(\ln P_{1q3,2q3}+\ln P_{1q4,2q4})\right]$$
$$+5\left[\frac{1}{2}(\ln P_{1q1,2q1}+\ln P_{1q2,2q2})\right]\Bigg\}. \tag{11}$$

Here the reconciled index comparing the first half of year 1 with the first half of year 2 is again written as a weighted geometric mean of competing unreconciled indices answering the same question. The unreconciled direct annualized biannual index $P_{11,21}$ gets a weight of 10/21, while the indirect index combining $P_{1q1,2q1}$ and $P_{1q2,2q2}$ gets a weight of 5/21, which is half that of the direct index. Thus far the pattern is analogous to the cases considered above. However, there are two more indirect indexes that also contribute to the solution for the reconciled index. These are the indirect indexes formed by combining $P_{1,2}$ and $P_{11,12}$, which gets a weight of 4/21, and the indirect index combining $P_{1,2}$, $P_{1q3,2q3}$ and $P_{1q4,2q4}$, which gets a weight of 2/21.

The solution for $P_{1q1,2q1}$ is:

$$\ln \hat{P}_{1q1,2q1} = \frac{1}{21}\{12\ln P_{1,2}+10\ln P_{11,21}-4\ln P_{12,22}+13\ln P_{1q1,2q1}-8\ln P_{1q2,2q2}$$
$$-\ln P_{1q3,2q3}-\ln P_{1q4,2q4}.\} \tag{12}$$

This solution can likewise be reinterpreted as a weighted geometric mean of competing unreconciled indices answering the same question:

$$\ln \hat{P}_{1q1,2q1} = \frac{1}{21}\{13\ln P_{1q1,2q1}+5(2\ln P_{11,21}-\ln P_{1q2,2q2})+2(4\ln P_{1,2}-2\ln P_{12,22}-\ln P_{1q2,2q2})$$
$$+(4\ln P_{1,2}-\ln P_{1q2,2q2}-\ln P_{1q3,2q3}-\ln P_{1q4,2q4})\}. \tag{13}$$

Now the unreconciled direct annualized quarterly index $P_{1q1,2q1}$ gets a weight of 13/21, the indirect index formed by combining $P_{11,21}$ and $P_{1q2,2q2}$ gets a weight of 5/21, the indirect index formed by combining $P_{1,2}$, $P_{12,22}$, and $P_{1q2,2q2}$ gets a weight of 2/21, and the indirect index combining $P_{1,2}$, $P_{1q2,2q2}$, $P_{1q3,2q3}$ and $P_{1q4,2q4}$ gets a weight of 1/21.

To discern the underlying structure it is useful to note that there are seven unknowns to be estimated: $\hat{P}_{1q1,2q1}$, $\hat{P}_{1q2,2q2}$, $\hat{P}_{1q3,2q3}$, $\hat{P}_{1q4,2q4}$, $\hat{P}_{11,21}$, $\hat{P}_{12,22}$, $\hat{P}_{1,2}$. In addition there are three constraints relating these seven unknowns, given by the reconciliation equations (i), (ii) and (iii) above. Hence there are only four degrees of freedom. For example, given values for $\hat{P}_{1q1,2q1}$, $\hat{P}_{1q2,2q2}$, $\hat{P}_{1q3,2q3}$, and $\hat{P}_{1q4,2q4}$, then $\hat{P}_{1,2}$, $\hat{P}_{11,21}$ and $\hat{P}_{12,22}$ can be derived from equations (i), (ii) and (iii), respectively.

There are 35 possible ways of combining four out of seven variables (i.e., $7!/(3!\times4!)$. Of these only 21 retain the four degrees of freedom (i.e., they do do not contain redundancies). For example, equations (v) and (vi) imply that the combinations $(\hat{P}_{1,2}, \hat{P}_{1q1,2q1}, \hat{P}_{1q2,2q2}, \hat{P}_{12,22})$, and $(\hat{P}_{1,2}, \hat{P}_{11,21}, \hat{P}_{1q3,2q3}, \hat{P}_{1q4,2q4})$ each have only three degrees of freedom, and hence cannot recover values for the missing three variables.

The 21 combinations of four variables that are sufficient to derive the other variables are listed below:

1. $P_{1q1,2q1}, P_{1q2,2q2}, P_{1q3,2q3}, P_{1q4,2q4}$
2. $P_{11,21}, P_{1q2,2q2}, P_{1q3,2q3}, P_{1q4,2q4}$
3. $P_{11,21}, P_{1q1,2q1}, P_{1q3,2q3}, P_{1q4,2q4}$
4. $P_{12,22}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q4,2q4}$
5. $P_{12,22}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q3,2q3}$
6. $P_{11,21}, P_{12,22}, P_{1q1,2q1}, P_{1q3,2q3}$
7. $P_{11,21}, P_{12,22}, P_{1q1,2q1}, P_{1q4,2q4}$
8. $P_{11,21}, P_{12,22}, P_{1q2,2q2}, P_{1q3,2q3}$
9. $P_{11,21}, P_{12,22}, P_{1q2,2q2}, P_{1q4,2q4}$
10. $P_{1,2}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q3,2q3}$
11. $P_{1,2}, P_{1q1,2q1}, P_{1q2,2q2}, P_{1q4,2q4}$
12. $P_{1,2}, P_{1q1,2q1}, P_{1q3,2q3}, P_{1q4,2q4}$
13. $P_{1,2}, P_{1q2,2q2}, P_{1q3,2q3}, P_{1q4,2q4}$
14. $P_{1,2}, P_{11,21}, P_{1q1,2q1}, P_{1q3,2q3}$
15. $P_{1,2}, P_{11,21}, P_{1q1,2q1}, P_{1q4,2q4}$

16. $P_{1,2}, P_{11,21}, P_{1q2,2q2}, P_{1q3,2q3}$
17. $P_{1,2}, P_{11,21}, P_{1q2,2q2}, P_{1q4,2q4}$
18. $P_{1,2}, P_{12,22}, P_{1q1,2q1}, P_{1q3,2q3}$
19. $P_{1,2}, P_{12,22}, P_{1q1,2q1}, P_{1q4,2q4}$
20. $P_{1,2}, P_{12,22}, P_{1q2,2q2}, P_{1q3,2q3}$
21. $P_{1,2}, P_{12,22}, P_{1q2,2q2}, P_{1q4,2q4}$

Each of these combinations provides a different way of constructing temporally reconciled indices from four unreconciled indices. The solution to the least squares reconciliation problem can be interpreted as the geometric mean of these 21 combinations. For example, consider the case of $\ln \hat{P}_{1q1,2q1}$. As noted there are only four chaining paths for estimating $\ln \hat{P}_{1q1,2q1}$ given the available indices:

A. $\ln P_{1q1,2q1}$

B. $2 \ln P_{11,21} - \ln P_{1q2,2q2}$

C. $4 \ln P_{1,2} - 2 \ln P_{12,22} - \ln P_{1q2,2q2}$

D. $4 \ln P_{1,2} - \ln P_{1q2,2q2} - \ln P_{1q3,2q3} - \ln P_{1q4,2q4}$

The combinations that include each of these chain paths are as follows:

A. 1, 3, 4, 5, 6, 7, 10, 11, 12, 14, 15, 18, 19

B. 2, 8, 9, 16, 17

C. 20, 21

D. 13

Thus it can readily be seen that the A approach appears 13 times, the B approach 5 times, C 2 times and D just 1 time. These weights correspond exactly with those in (13).

4 An Analogy with the Multilateral Price Index Literature

There is an interesting parallel here with the GEKS method used to transitivize price indices in the multilateral price index literature (see for example Diewert 1999 and Balk 2008). In the GEKS context there are no temporal hierarchies. Rather, the GEKS method takes a set of intransitive bilateral price indices and alters them by the logarithmic least squares amount necessary to make them transitive (or reconciled using our terminology).

Algebraically, this least squares problem can be written as follows:

$$\min_{\ln P_j, \ln P_k} \sum_{j=1}^{I}\sum_{k=1}^{I}(\ln P_k - \ln P_j - \ln P_{j,k})^2, \tag{14}$$

where I is the number of countries participating in a multilateral comparison, $P_{j,k}$ denotes the observed bilateral price index comparison between countries j and k, P_k denotes a multilateral (reconciled) price index for country k, and the normalization $P_1 = 1$ is imposed. The solutions, $\hat{\ln P_j}, \hat{\ln P_k}$ are the ordinary least squares (OLS) estimators of $\ln P_j, \ln P_k$ in the regression model:

$$\ln P_{j,k} = \ln P_k - \ln P_j + \epsilon_{j,k}, \tag{15}$$

where $\epsilon_{j,k}$ is a random error term.

The GEKS price indices take the following form:

$$\frac{P_k^{GEKS}}{P_j^{GEKS}} = \exp\left(\hat{\ln P_k} - \hat{\ln P_j}\right) = \prod_{i=1}^{I}(P_{j,i} \times P_{i,k})^{1/I} = (P_{j,k})^{2/I}\prod_{i \neq j,k}^{I}(P_{j,i} \times P_{i,k})^{1/I}, \tag{16}$$

where P_k^{GEKS} denotes the GEKS price index for country k, and $i = 1, \ldots, I$ indices the countries included in the multilateral comparison.[1]

As can be seen from (16), the GEKS solution for a pair of countries j and k gives twice the weight to the direct bilateral comparison between j and k, as to all the indirect comparisons (each of which involves chaining via a third country i). This finding is reminiscent of our result for two layer hierarchies in (4). An intriguing parallel also exists with the result for our three layer hierarchy in (9). Here the reconciled annual comparison gives the indirect comparison made using biannual indices half the weight as the direct comparison, while the indirect comparison using quarterly indices gets quarter the weight.

5 Weighted Reconciliation in Three Layer Hierarchies

Weighting by frequency

We consider here two alternative ways of introducing weights into the model. The first is to use weighted least squares to give different weights to different frequencies. In this case, the reconciled indices are estimated as follows:

$$\hat{y} = S\tilde{\beta} = S(S'WS)^{-1}S'Wy, \tag{17}$$

[1] We have assumed in (16) that the bilateral price index formula $P_{j,k}$ satisfies the country reversal test (i.e., $P_{j,k} = 1/P_{k,j}$). All superlative price indices satisfy this test (see Diewert 1976).

where W is a diagonal weights matrix. Focusing on the case of the three-layer hierarchy of annual, biannual and quarterly indices, we consider two examples of weights matrices below:

$$W_1 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

These weights matrices generate the following projection matrices:

$$S(S'W_1S)^{-1}S'W_1 = \frac{1}{21}\begin{pmatrix} 16 & 4 & 4 & 1 & 1 & 1 & 1 \\ 8 & 10.4 & -6.4 & 2.6 & 2.6 & -1.6 & -1.6 \\ 8 & -6.4 & 10.4 & -1.6 & -1.6 & 2.6 & 2.6 \\ 4 & 5.2 & -3.2 & 11.8 & -9.2 & -0.8 & -0.8 \\ 4 & 5.2 & -3.2 & -9.2 & 11.8 & -0.8 & -0.8 \\ 4 & -3.2 & 5.2 & -0.8 & -0.8 & 11.8 & -9.2 \\ 4 & -3.2 & 5.2 & -0.8 & -0.8 & -9.2 & 11.8 \end{pmatrix}, \tag{19}$$

$$S(S'W_2S)^{-1}S'W_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

The solution for the reconciled price index $\hat{P}_{1,2}$ obtained from the W_1 weights matrix can be written as follows:

$$\ln \hat{P}_{1,2} = \frac{1}{21}\left\{16 \ln P_{1,2} + 4\left[\frac{1}{2}(\ln P_{11,21} + \ln P_{12,22})\right]\right.$$
$$\left. + \left[\frac{1}{4}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2} + \ln P_{1q3,2q3} + \ln P_{1q4,2q4})\right]\right\}. \tag{21}$$

Comparing (9) and (21) it can be seen that now the unreconciled price index $P_{1,2}$ is given more weight in determining the reconciled indices. By contrast, the weights matrix W_2 gives zero weight to the higher frequency indices. Now the reconciled lower frequency indices are determined purely by the highest frequency index:

$$\ln \hat{P}_{1,2} = \frac{1}{4}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2} + \ln P_{1q3,2q3} + \ln P_{1q4,2q4}),$$

$$\ln \hat{P}_{11,21} = \frac{1}{2}(\ln P_{1q1,2q1} + \ln P_{1q2,2q2}),$$
$$\ln \hat{P}_{12,22} = \frac{1}{2}(\ln P_{1q3,2q3} + \ln P_{1q4,2q4}).$$

This approach, therefore, gives users the flexibility to adjust the weights on indices of different frequencies as desired.

Weighting by number of observations

When information is available on the number of observations over which an index was constructed, this information can be used in the reconciliation process. For example, the empirical comparison in section 6 uses focuses on house price indices for Sydney, Australia. There is a strong seasonal cycle in the number of transactions in Sydney. Very few transactions occur in the period from the second half of December through to the end of February. In consequence the number of transactions is always lower in the first quarter of the year. The effect is even more dramatic when the highest frequency index is monthly.

Focusing on the quarterly case, an implication of the seasonal cycle in transactions is that the annualized quarterly index $P_{1q1,2q1}$ may tend to be less reliable than the other annualized quarterly indices $P_{1q2,2q2}$, $P_{1q3,2q3}$ and $P_{1q4,2q4}$. Now let n_{1q1} denote the number of transactions in year 1, quarter 1, n_{11} the number of transactions in the first half of year 1, and n_1 the number of transactions in year 1. Transaction weights can now be introduced by altering the y vector as follows:

$$y = \begin{pmatrix} \ln P_{1,2} \\ w_{11,21}(\ln P_{11,21}) \\ w_{12,22}(\ln P_{12,22}) \\ w_{1q1,2q1}(\ln P_{1q1,2q1}) \\ w_{1q2,2q2}(\ln P_{1q2,2q2}) \\ w_{1q3,2q3}(\ln P_{1q3,2q3}) \\ w_{1q4,2q4}(\ln P_{1q4,2q4}) \end{pmatrix}, \tag{22}$$

where

$$w_{11,21} = \frac{n_{11} + n_{21}}{n_1 + n_2}, \quad w_{12,22} = \frac{n_{12} + n_{22}}{n_1 + n_2},$$
$$w_{1q1,2q1} = \frac{n_{1q1} + n_{2q1}}{n_1 + n_2}, \quad w_{1q2,2q2} = \frac{n_{1q2} + n_{2q2}}{n_1 + n_2},$$
$$w_{1q3,2q3} = \frac{n_{1q3} + n_{2q3}}{n_1 + n_2}, \quad w_{1q4,2q4} = \frac{n_{1q4} + n_{2q4}}{n_1 + n_2}.$$

6 An Application to House Price Indices

In this section we present an example of a reconciliation of price indices for residential housing. We consider two price indices, both are hedonic imputed indices; however, they differ in the underlying model. The first controls for location using a geospline (Model G) and the second using postcode dummies (Model PC). These were both considered in Hill et al (2018). The models are estimated with data from Sydney, Australia covering the period 2001-2014 at each of the four frequencies, annual, quarterly, monthly and bimonthly. At the annual frequency the models are estimated for each year separately. Model G is a semi-parametric generalised additive model (indices will be labelled GAM), Model PC is a hedonic regression with postcode dummies estimated by least squares (indices will be labelled PC). For the higher frequencies (quarterly, monthly and bimonthly) the models are written is state-space form and estimated using Kalman filtering techniques. The indices will be labelled SSG if Model G state-space is used and SSPC if Model PC state-space is used. Details of how these models are estimated can be found in Hill et al (2018).

We use the notation "_U" to denote an index that has not been reconciled, and "_R" for an index that has been reconciled without using any of the weighted alternatives discussed in Section 5. We will denote by "_R(W1)", "_R(W2)" and "_R(TW)" the reconciled indices obtained using weightings $W1$, $W2$ (see equation 18) and the transaction weighted y (see equation 22), respectively.

6.1 The Annual, Quarterly, Monthly, Bimonthly Hierarchy

It is easy to verify that S and y (in the non-transaction weighted case) are defined as follows in this four-level hierarchy:

$$S = \begin{bmatrix} \mathbf{j}'_{24} \\ \mathbf{I}_4 \otimes \mathbf{j}'_6 \\ \mathbf{I}_{12} \otimes \mathbf{j}'_2 \\ \mathbf{I}_{24} \end{bmatrix} \qquad y = \begin{pmatrix} \ln P_{1,2} \\ ((1/4) \ln P_{1q1,2q1} \\ \vdots \\ (1/4) \ln P_{1q4,2q4} \\ (1/12) \ln P_{1m1,2m1} \\ (1/12) \ln P_{1m2,2m2} \\ \vdots \\ (1/12) \ln P_{1m11,2m11} \\ (1/12) \ln P_{1m12,2m12} \\ (1/24) \ln P_{1bm1,2bm1} \\ (1/24) \ln P_{1bm2,2bm2} \\ \vdots \\ (1/24) \ln P_{1bm23,2bm23} \\ (1/24) \ln P_{1bm24,2bm24} \end{pmatrix}.$$

where,

$\mathbf{j}'_m$ is an m row vector of 1's

$\mathbf{I}_m$ is an identity of size m

and $\otimes$ is the Kronecker tensor product[2]

When the system is specified with transaction weights y (see equation 22), these replace the fixed weights of 1/4, 1/12 and 1/24, and change overtime. Figure 1, shows plots of the number of transactions per time period at each of the four frequencies.

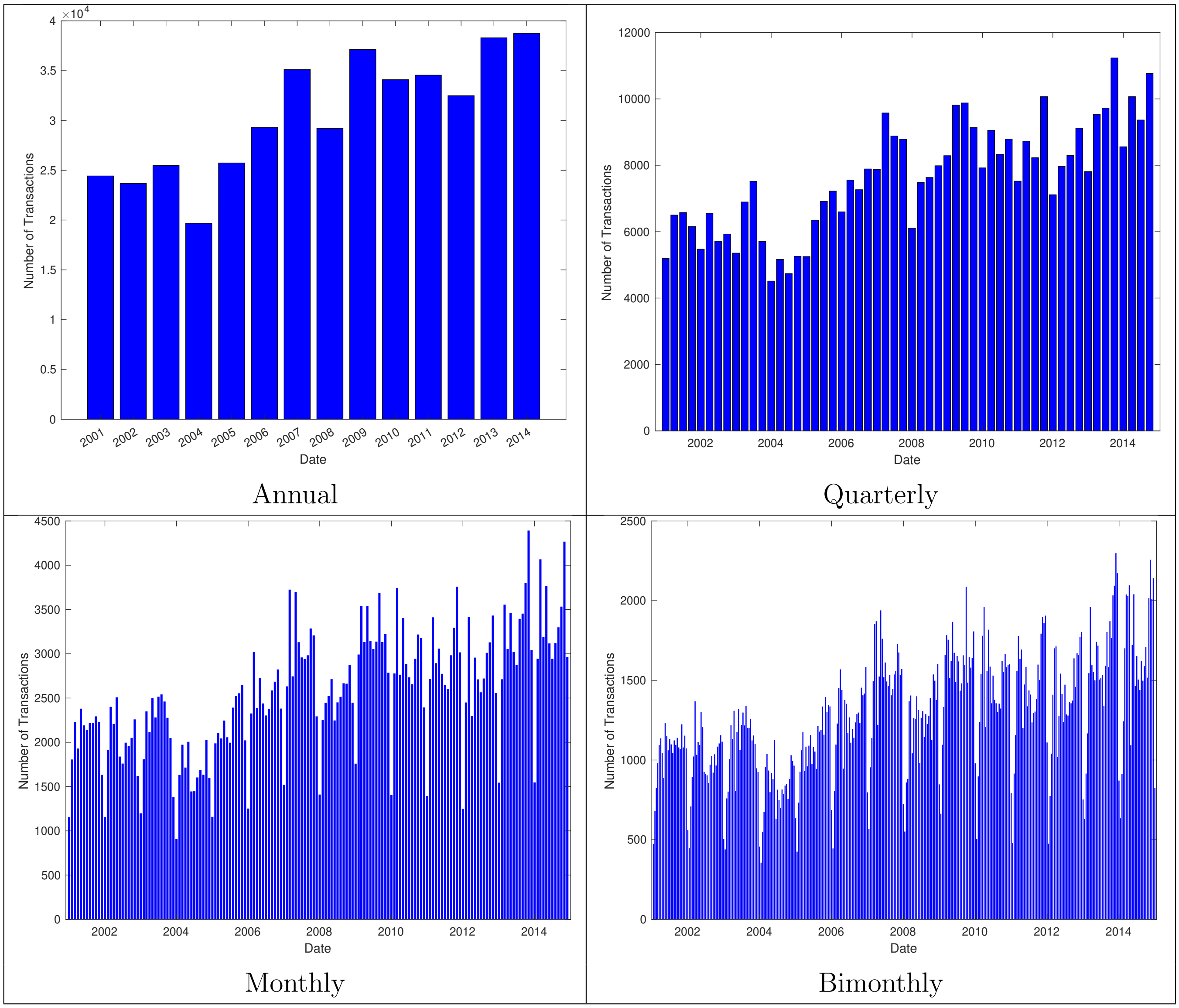

Figure 1: Number of Transactions per period at each frequency

Table 1 presents the unreconciled and reconciled annual indices from Model G.

[2]The Kronecker tensor product of matrices A and B is an $m \times p - by - n \times q$ matrix formed by taking all possible products between the elements of A (an $m - by - n$ matrix) and B (a $p - by - q$ matrix)

Table 2 presents the corresponding annual indices from Model PC. Figures 2 and 3 present plots of the unreconciled and all reconciled indices from each Model, G and PC, respectively. The figures are presented in four panels with one for each frequency. In both cases, we provide the computed year-on-year price change at the corresponding frequency. This index, labelled "MedIndex" in the tables and "Median Y-oY Index" in the plots, is provided to serve as a benchmark, as this is a quality and location unadjusted index.

Table 1: Annual Unreconciled and Reconciled Generalised Additive Model based Year-on-Year Indices

Year	GAM_U	GAM_R	GAM_R(W1)	GAM_R(W2)	GAM_R(TW)	MedIndex
2001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2002	1.2229	1.2243	1.2233	1.2270	1.2249	1.0000
2003	1.1615	1.1490	1.1583	1.1158	1.1483	1.0000
2004	1.0385	1.0522	1.0420	1.0910	1.0519	1.0875
2005	0.9663	0.9611	0.9649	0.9485	0.9608	0.9366
2006	0.9988	0.9965	0.9982	0.9866	0.9970	0.9818
2007	1.0453	1.0442	1.0450	1.0417	1.0441	1.0556
2008	1.0023	0.9901	0.9991	0.9624	0.9895	0.9561
2009	1.0518	1.0487	1.0511	1.0352	1.0500	1.0367
2010	1.1027	1.1137	1.1055	1.1428	1.1119	1.2008
2011	1.0024	0.9888	0.9990	0.9525	0.9887	0.9403
2012	1.0095	1.0185	1.0117	1.0451	1.0188	1.0815
2013	1.0989	1.0953	1.0980	1.0840	1.0960	1.0957
2014	1.1687	1.1626	1.1672	1.1422	1.1606	1.1409

Note: Unreconciled Index is from the Generalised Additive Model (GAM) estimated for each year separately. Reconciled Indices are obtained using the GAM_U and SSG_U for Quarterly, Monthly and Bimonthly Models

Table 2: Annual Unreconciled and Reconciled Postcodes Dummies Model based Year-on-Year Indices

Year	PC_U	PC_R	PC_R(W1)	PC_R(W2)	PC_R(TW)	MedIndex
2001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2002	1.2240	1.2230	1.2240	1.2210	1.2240	1.0000
2003	1.1590	1.1590	1.1590	1.1600	1.1590	1.0000
2004	1.0390	1.0400	1.0390	1.0450	1.0390	1.0875
2005	0.9670	0.9670	0.9670	0.9680	0.9670	0.9366
2006	0.9990	0.9990	0.9990	1.0000	0.9990	0.9818
2007	1.0430	1.0430	1.0430	1.0430	1.0430	1.0556
2008	1.0020	1.0030	1.0020	1.0060	1.0020	0.9561
2009	1.0520	1.0490	1.0510	1.0440	1.0500	1.0367
2010	1.1030	1.1030	1.1030	1.1020	1.1020	1.2008
2011	1.0020	1.0020	1.0020	1.0020	1.0020	0.9403
2012	1.0080	1.0080	1.0080	1.0060	1.0080	1.0815
2013	1.0990	1.0980	1.0980	1.0940	1.0980	1.0957
2014	1.1690	1.1690	1.1690	1.1690	1.1670	1.1409

Note: Unreconciled Index is from the Postcodes Dummies Models estimated by OLS for each year separately (PC). Reconciled Indices are obtained using the PC_U and SSPC_U for Quarterly, Monthly and Bimonthly Models

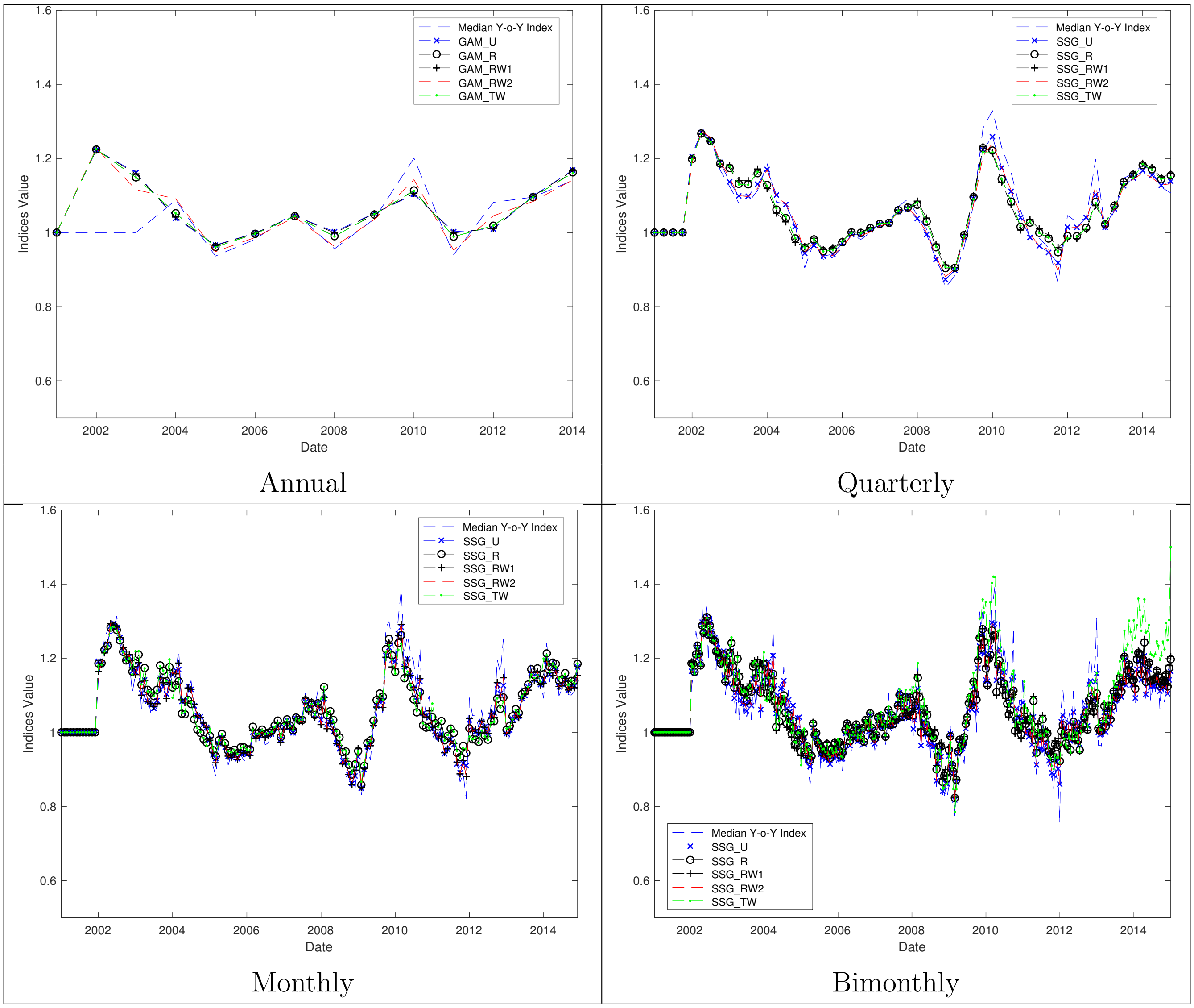

Figure 2: SSG Year-on-Year Indices - Unreconciled and Reconciled and Median Prices Index

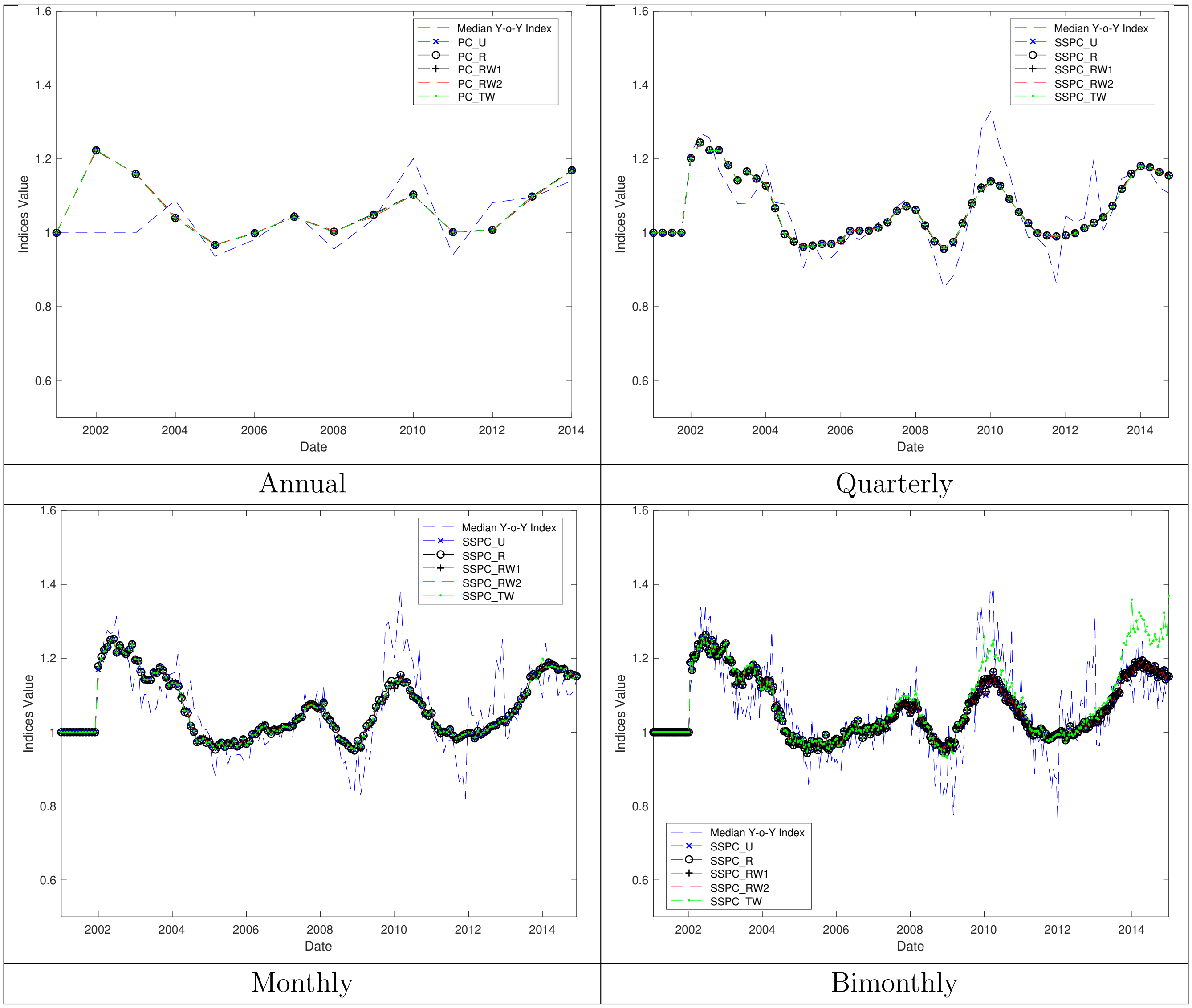

Figure 3: SSPC Year-on-Year Indices - Unreconciled and Reconciled and Median Prices Index

6.2 The Benefits of Temporal Reconciliation

The main benefit of temporal reconciliation is to provide a set of statistics that are consistent in aggregation across temporal frequencies. In this section we explore other possible benefits. We study whether the temporal reconciliation approaches can reduce the volatility of the high frequency indexes. If this is the case, we might obtain more precise estimates of turning points which have benefits to those interested in understanding and responding to changes in the cycle.

There are potentially a number of ways of quantifying the extent to which temporal reconciliation leads to a series with greater informational content than the unreconciled index. Perhaps the most straightforward approach would be to calculate the variance of the reconciled and unreconciled series. This is not unreasonable but one problem is that the mean or true rate of price change will vary over the period examined rather than being a constant. In fact, there is no reason to contemplate that these indices are mean reverting in theory. In practice, standard tests for non-stationarity for the higher frequency series cannot reject non-stationarity. Thus, any statistic based on second moments cannot be used as they do not exist. This is a standard problem in the econometric analysis of financial time series. To compare the variance of the stationary part of the series, i.e. its volatility, the measurement is based on the variances of the log first differences. These are labelled "returns" in the financial time series literature. Returns are stationary and thus standard statistics can be computed on these. We compute the variance-covariance of the returns of the unreconciled, reconciled indices and the median price change. This is done separately for the indices from the G and PC models. We compute two types of returns, a first difference and a first seasonal difference. As the computed indices are year-on-year, the second preserves the structure. Equations 23 and 24 define the construction of the returns.

$$Ret_{t,t-1} = log(Index_t) - log(Index_{t-1}) \tag{23}$$

$$Ret_{t,t-s} = log(Index_t) - log(Index_{t-s}) \tag{24}$$

where,
$s = 24, 12, 4$
$Index$ is a given index from the SSG or SSPC type

Tables 3-6 present the computed variance-covariance of the returns (as defined in (24)) of indices at annual, quarterly, monthly and bimonthly frequencies. Within each table the variance-covariance estimates for indices from Model G and PC are shown separately.

Table 3: Variance-Covariance of Annual Indices's Returns (log first differences)

Annual Variance-Covariance Estimates					
GAM_U	GAM_R	GAM_R(W1)	GAM_R(W2)	GAM_R(TW)	MedIndex
0.0074	0.0076	0.0074	0.0081	0.0076	0.0040
0.0076	**0.0080**	0.0077	0.0091	0.0080	0.0054
0.0074	0.0077	**0.0075**	0.0083	0.0077	0.0043
0.0081	0.0091	0.0083	**0.0118**	0.0091	0.0093
0.0076	0.0080	0.0077	0.0091	**0.0080**	0.0054
0.0040	0.0054	0.0043	0.0093	0.0054	**0.0129**
PC_U	PC_R	PC_R(W1)	PC_R(W2)	PC_R(TW)	MEdIndex
0.0074	0.0074	0.0074	0.0073	0.0074	0.0040
0.0074	**0.0073**	0.0074	0.0073	0.0073	0.0040
0.0074	0.0074	**0.0074**	0.0073	0.0074	0.0040
0.0073	0.0073	0.0073	**0.0072**	0.0073	0.0040
0.0074	0.0073	0.0074	0.0073	**0.0074**	0.0039
0.0040	0.0040	0.0040	0.0040	0.0039	**0.0129**
Note: The diagonal elements are variance estimates, off-diagonal are covariance estimates					

Table 4: Variance-Covariance of Quarterly Year-on-Year Indices's Returns (log seasonal differences)

Quarterly Variance-Covariance Estimates					
SSG_U	SSG_R	SSG_R(W1)	SSG_R(W2)	SSG_R(TW)	MedIndex
0.0185	0.0158	0.0151	0.0180	0.0156	0.0215
0.0158	**0.0146**	0.0144	0.0154	0.0144	0.0181
0.0151	0.0144	**0.0142**	0.0147	0.0142	0.0172
0.0180	0.0154	0.0147	**0.0177**	0.0152	0.0212
0.0156	0.0144	0.0142	0.0152	**0.0143**	0.0179
0.0215	0.0181	0.0172	0.0212	0.0179	**0.0261**
SSPC_U	SSPC_R	SSPC_R(W1)	SSPC_R(W2)	SSPC_R(TW)	MEdIndex
0.0093	0.0093	0.0093	0.0092	0.0092	0.0126
0.0093	**0.0094**	0.0094	0.0092	0.0092	0.0126
0.0093	0.0094	**0.0094**	0.0092	0.0093	0.0126
0.0092	0.0092	0.0092	**0.0091**	0.0091	0.0124
0.0092	0.0092	0.0093	0.0091	**0.0091**	0.0124
0.0126	0.0126	0.0126	0.0124	0.0124	**0.0261**
Note: The diagonal elements are variance estimates, off-diagonal are covariance estimates					

Table 6: Variance-Covariance of Bimonthly Year-on-Year Indices's Returns (log seasonal differences)

Bimonthly Variance-Covariance Estimates					
SSG_U	SSG_R	SSG_R(W1)	SSG_R(W2)	SSG_R(TW)	MedIndex
0.0198	0.0174	0.0167	0.0198	0.0210	0.0232
0.0174	**0.0166**	0.0163	0.0174	0.0201	0.0200
0.0167	0.0163	**0.0162**	0.0167	0.0198	0.0191
0.0198	0.0174	0.0167	**0.0198**	0.0210	0.0232
0.0210	0.0201	0.0198	0.0210	**0.0254**	0.0244
0.0232	0.0200	0.0191	0.0232	0.0244	**0.0294**
SSPC_U	SSPC_R	SSPC_R(W1)	SSPC_R(W2)	SSPC_R(TW)	MEdIndex
0.0094	0.0095	0.0095	0.0094	0.0111	0.0122
0.0095	**0.0096**	0.0096	0.0095	0.0112	0.0124
0.0095	0.0096	**0.0096**	0.0095	0.0112	0.0124
0.0094	0.0095	0.0095	**0.0094**	0.0111	0.0122
0.0111	0.0112	0.0112	0.0111	**0.0137**	0.0151
0.0122	0.0124	0.0124	0.0122	0.0151	**0.0294**
Note: The diagonal elements are variance estimates, off-diagonal are covariance estimates					

Table 5: Variance-Covariance of Monthly Indices's Returns (log seasonal differences)

Monthly Variance-Covariance Estimates					
SSG_U	SSG_R	SSG_R(W1)	SSG_R(W2)	SSG_R(TW)	MedIndex
0.0189	0.0165	0.0188	0.0188	0.0160	0.0220
0.0165	**0.0156**	0.0164	0.0164	0.0151	0.0189
0.0188	0.0164	**0.0188**	0.0188	0.0158	0.0221
0.0188	0.0164	0.0188	**0.0188**	0.0158	0.0221
0.0160	0.0151	0.0158	0.0158	**0.0149**	0.0182
0.0220	0.0189	0.0221	0.0221	0.0182	**0.0273**
SSPC_U	SSPC_R	SSPC_R(W1)	SSPC_R(W2)	SSPC_R(TW)	MEdIndex
0.0093	0.0094	0.0093	0.0093	0.0093	0.0123
0.0094	**0.0095**	0.0094	0.0094	0.0093	0.0123
0.0093	0.0094	**0.0093**	0.0093	0.0092	0.0122
0.0093	0.0094	0.0093	**0.0093**	0.0092	0.0122
0.0093	0.0093	0.0092	0.0092	**0.0092**	0.0121
0.0123	0.0123	0.0122	0.0122	0.0121	**0.0273**
Note: The diagonal elements are variance estimates, off-diagonal are covariance estimates					

6.3 Discussion of Findings

The empirical results presented above are preliminary; however, there are some interesting findings:

- All indices are smoother than the crude Median Index. The annual price change estimated by the indices between 2001 and 2002, and 2002 and 2003 is very different from that indicated by the median change. The median indicates prices did not change over this period, while both sets of price indices estimate changes in the order of 22% and 16%, respectively.
- The reconciled indices, either unweighted ($\_R$) or weighted under $W1$ or $W2$ do not appear to change the level of the unreconciled indices at the higher frequencies, monthly and bimonthly. There is some disagreement at the quarterly frequency for the SSG case where all reconciled indices show the turn in the cycle to be in an earlier quarter than that shown by the unreconciled index and the median. This is not the case for the SSPC quarterly index.
- The SSPC indices are much smoother than the SSG type indices. Hill et al (2018) found the SSG model (at the weekly frequency) to outperform competitors significantly in predicting the observed price changes in properties that sold more than once over the sample. The reconciliation does not appear to change the indices levels, except for the one case discussed next.
- The use of transaction weights ($\_TW$) to transform the log indices seems to produce reconciled indices that are mostly in agreement with the others, except in the bimonthly frequency. For both (SSG and SSPC) indices there are two episodes which show this reconciled version of the indices deviating from the others. The period covering late 2009 and first half of 2010, is a short lived recovery following the global financial crisis. The transactions weighted reconciled index shows much higher price changes for the period than that shown by all other indices. The median change for the year 2014 is in the neighbourhood of 10-15%, the unreconciled and reconciled SSPC indices indicate a change close to but below 20%, but the change estimated by the transactions weighted reconciled index is in the neighbourhood of 25-30%. The reconciled SSG indices are closer to each other; however, the transactions weighted index is clearly at a higher level than the others for 2014.

- The comparison of the volatility of unreconciled and reconciled indices does not yield a clear conclusion. All indices show a large reduction in volatility from that of the median index. The indices based on Model PC show no reduction in volatility between unreconciled and reconciled indices. The volatility of the SSG reconciled but unweighted ((SSG_R) is smaller at quarterly, monthly and bimonthly. The variance of the returns for the bimonthly transaction weighted reconciled indices, $SSG\_R(TW)$ $SSG\_R(TW)$ as much larger than that of all other indices with the exception of the median.

7 Conclusion

In this paper we extend an approach proposed in the literature to reconcile time series across frequencies to the case of price indices. In this case we have to relax the assumption that the time series at the top of the hierarchy is equal to the sum of the bottom time series in the hierarchy. Instead of assuming that the price indices in log form are additive over the temporal hierarchy, we impose different identifying restrictions. Our approach draws on ideas from the multilateral price index literature, and especially the Gini-Eltetö-Szulc (GEKS) method (see for example Diewert 1999 and Balk 2008). The key is to formulate different combinations of indices that provide alternative answers to the same question.

The paper considers the case of reconciling year-on-year indices at different frequencies and under a number of alternative weighting schemes. The simplest case of a temporal hierarchy of price indices is where there are two layers, and the higher frequency is double that of the lower frequency. The reconciliation is done at the level of the lowest frequency index. We reconcile each pair of adjacent years separately. We also propose three alternative reconciliations involving alternative weighting schemes.

A comparison with the multilateral price index literature is shown. The GEKS solution for a pair of countries j and k gives twice the weight to the direct bilateral comparison between j and k, as to all the indirect comparisons (each of which involves chaining via a third country i). This finding is reminiscent of our result for two layer hierarchies. An intriguing parallel also exists with the result for a three layer hierarchy. Here the reconciled annual comparison gives the indirect comparison made using biannual indices half the weight as the direct comparison, while the indirect comparison using quarterly indices gets quarter the weight.

The empirical example presents temporal hierarchies for hedonic property price indices in Sydney, Australia. The frequency of the indices considered are annual, quarterly, monthly, and bimonthly. The two latter frequencies are of special interest to central banks. For example, the ECB meets bimonthly, while the Reserve Bank of Australia meets monthly. Real time high frequency indices can provide a timely indication of the state of the real estate market. We consider hedonic imputed indices constructed from semilog hedonic models which control for property location using either postcode dummies or a non-parametric geo-spline, and construct volatility measures to compare the unreconciled and reconciled indices. We find only one of the four alternative reconciliation schemes changed the level of the unreconciled index during two periods of market volatility. Comparing the volatility of the unreconciled and reconciled indices does not yield a uniform and clear conclusion. However, the results presented are preliminary and should be taken as work in progress.

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